SINGLY EVEN MAGIC SQUARES (4 θ +2)

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Section 1 Outline

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Section 2 Introduction

Singly even magic squares are of the form $4\theta + 2$, where θ is a natural number.

I saw a solution to singly even magic squares belonging to Shin, Kwon Young at http://user.chollian.net/~brainstm/other.htm.

Using his solution as my starting point, I wanted to create a magic square that as closely as possible resembles the 1-2-1 pattern used by magic squares of the form 4θ. Interestingly magic squares of the form $4\theta + 2$ can be shown to have two separate 1-2-1 patterns.

The larger 1-2-1 pattern has its sections separated from each other by either a single row or column. The remaining 1-2-1 pattern involves the two rows and two columns that acted as separators for the sections in the larger 1-2-1 pattern. Since the diagonals in the remaining 2-row/2-column pattern are required to remain fixed, this requires an additional four exchanges to equalize the square. However, regardless of how large you make θ, only these four exchanges will be required. One exchange will be in the larger 1-2-1 pattern and three exchanges will be in the smaller 1-2-1 pattern.

Section 3   Detailed Explanation for the General Singly Even Magic Square

Magic Squares that have a side of $(4\theta + 2) = x$ units, where θ is a natural number, can be shown to follow the same 1-2-1 pattern as a magic square whose side is 4θ units. However, the sections using the 1-2-1 pattern are separated from each other by either a single row or column.

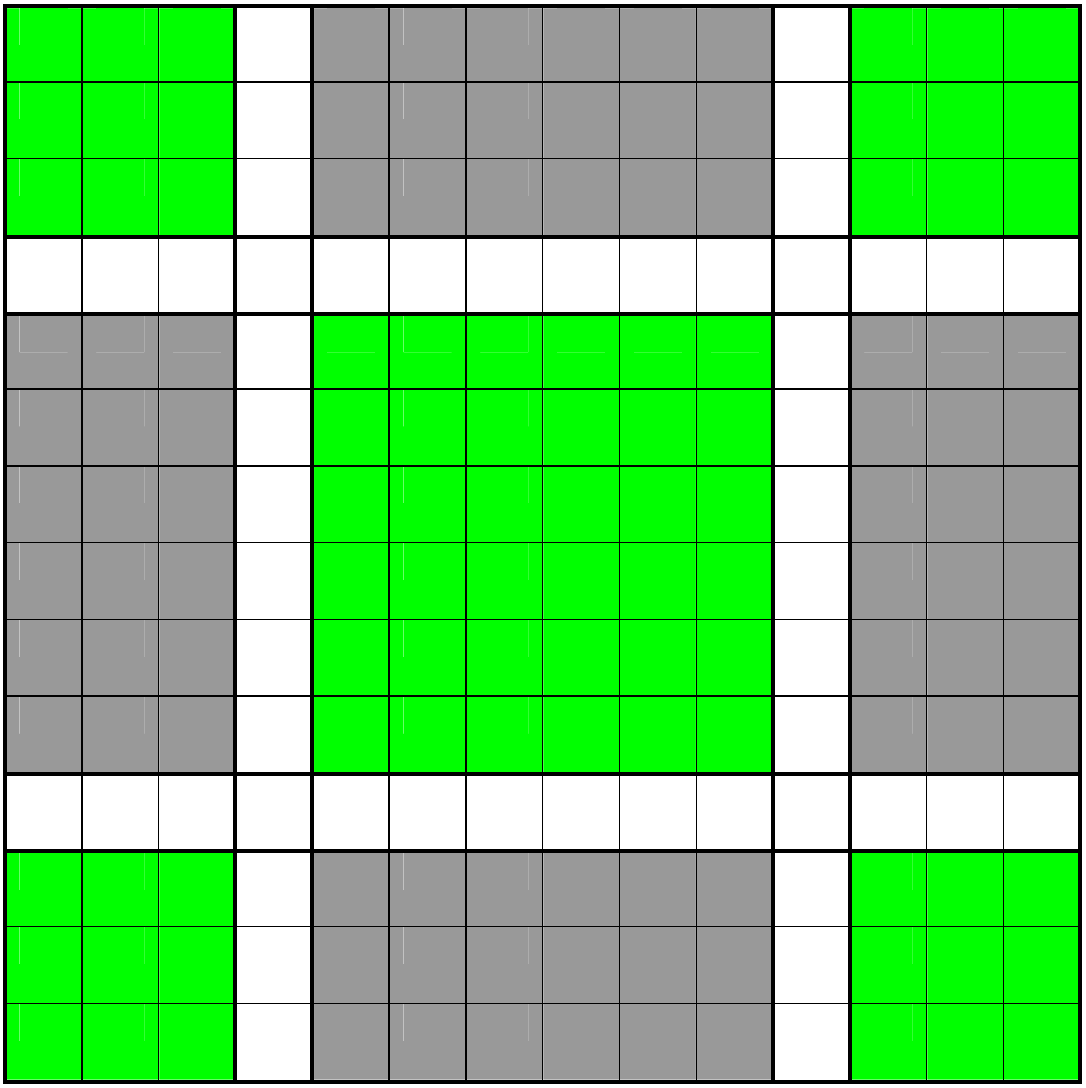
Interestingly, the remaining two rows and two columns minus the diagonal cells have their own 1-2-1 pattern.

There are $(4\theta + 2)^2 = x^2$ cells to fill, of which $(4\theta)^2 = (x-2)^2$ are occupied by the larger 1-2-1 pattern. Thus $(16\theta + 4) = (4x - 4)$ cells remain to be filled.

Requiring the remaining 4 values on the diagonals to keep their original values guarantees that the diagonals will add up to $\frac{1}{2}x(x^2+1)$. This is also in keeping with the 1-2-1 pattern where all diagonals are in their original position. This reduces the number of cells to fill to $4x-8$. However, this requirement will cause us to complete one additional column exchange and three additional row exchanges, involving two cells with each exchange.

All rows and columns must add up to $\frac{1}{2}x(x^2+1)$. All rows and columns that are only missing two elements, their remaining two values must sum to (x^2+1) because the rest of the blocks are filled using the 1-2-1 pattern.

Figure 3.1 Larger portion using 1-2-1 pattern



Green areas keep their original order; gray areas have their order reversed.

***** On the last step, we will need to swap two non-diagonal cells from the two center columns on the same row. Whether or not it will be from the green or gray area will depend on how we choose to fill the two remaining rows. The rest of the 1-2-1 area will remain the same.

Figure 3.2 Completion of diagonals – kept in original positions.

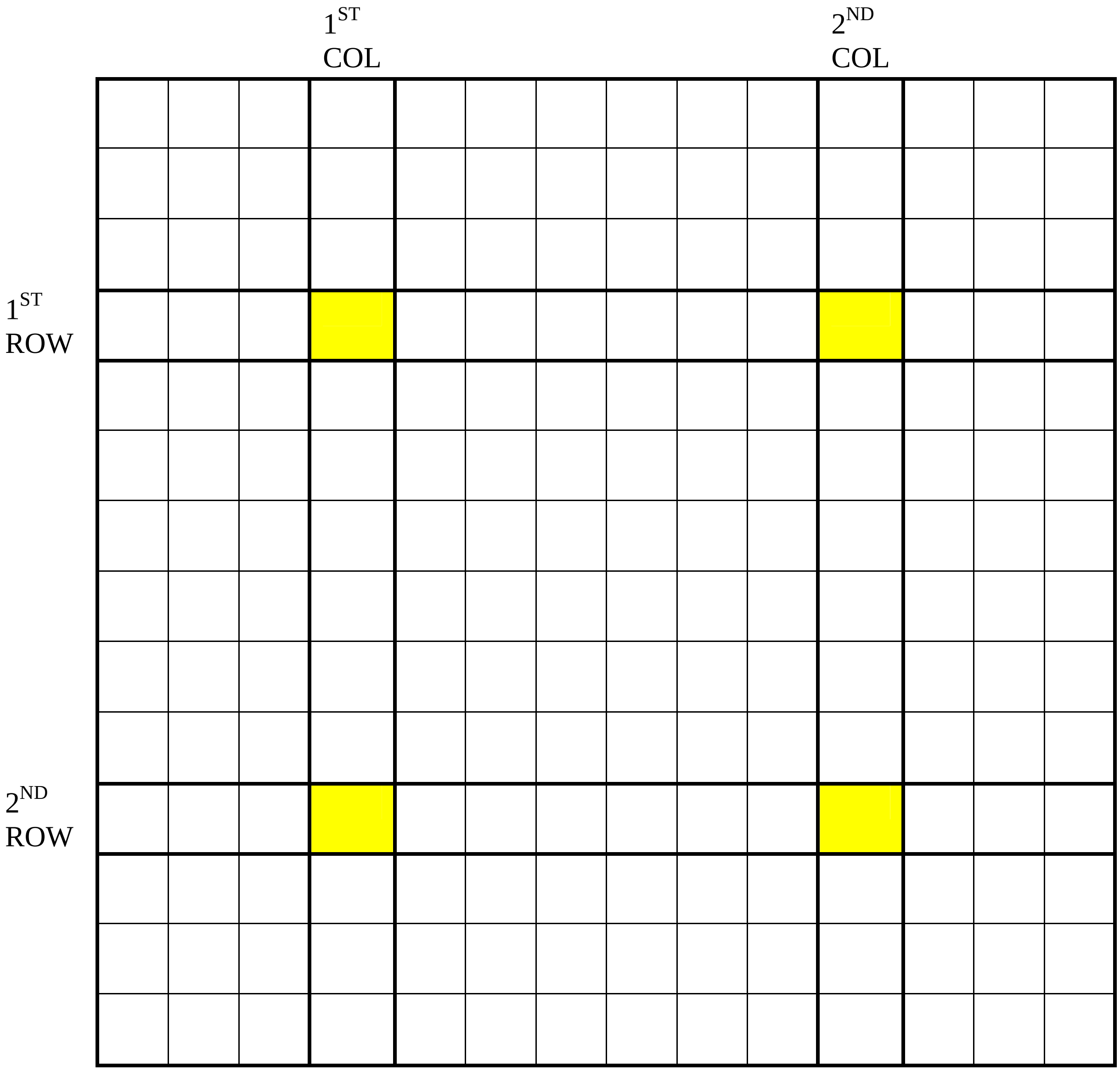
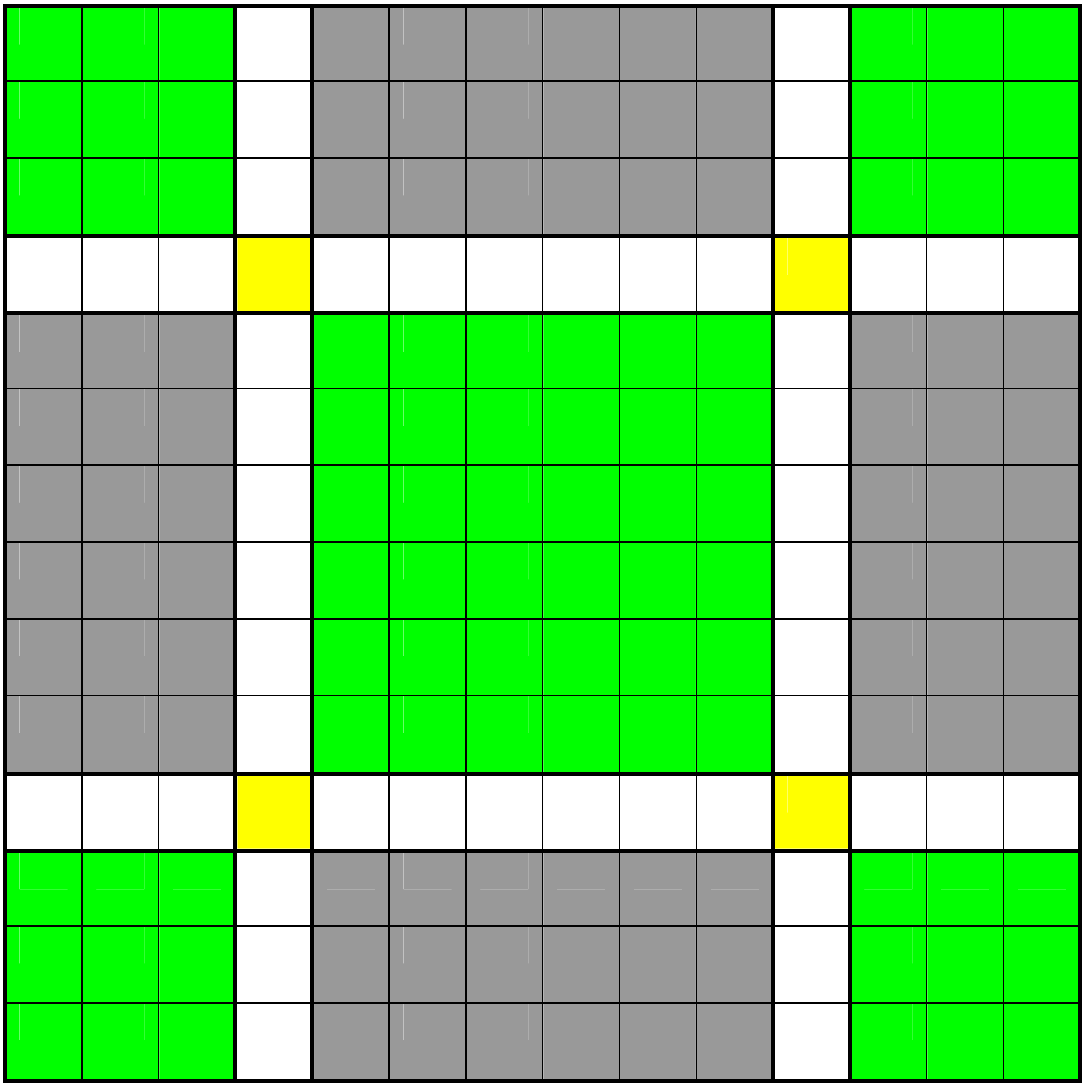
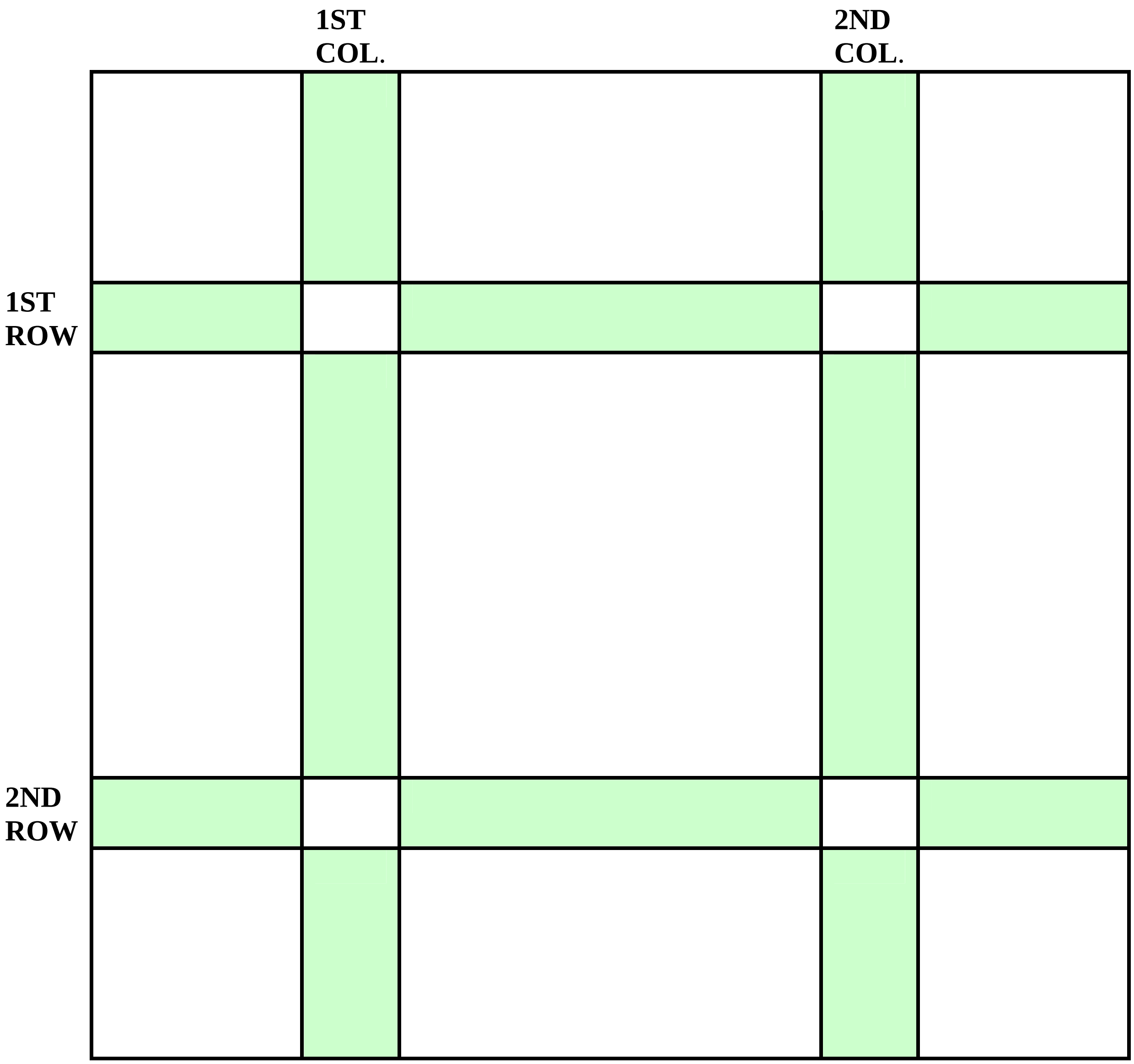


Figure 3.3 Previous two charts combined.



Green areas keep their original order; gray areas have their order reversed. Yellow areas are the four diagonal values kept in their original position.

Figure 3.4 Columns/rows involved in smaller 1-2-1 pattern.



The values below assume original placement.

Figure 3.5 Reverse order of 2^{nd} row and 2^{nd} column.

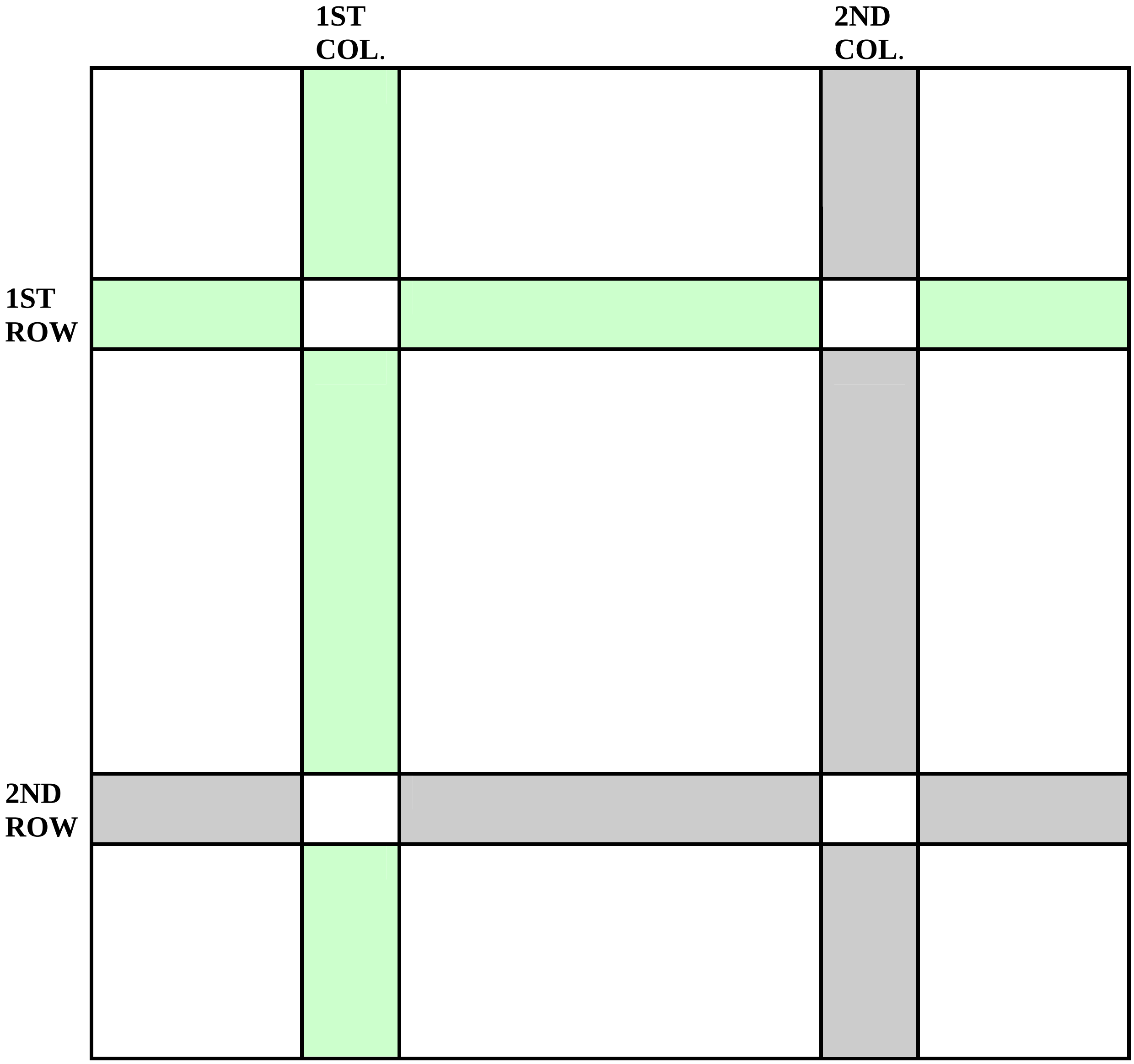


Figure 3.6: Swap ½ of the row and ½ of the column.

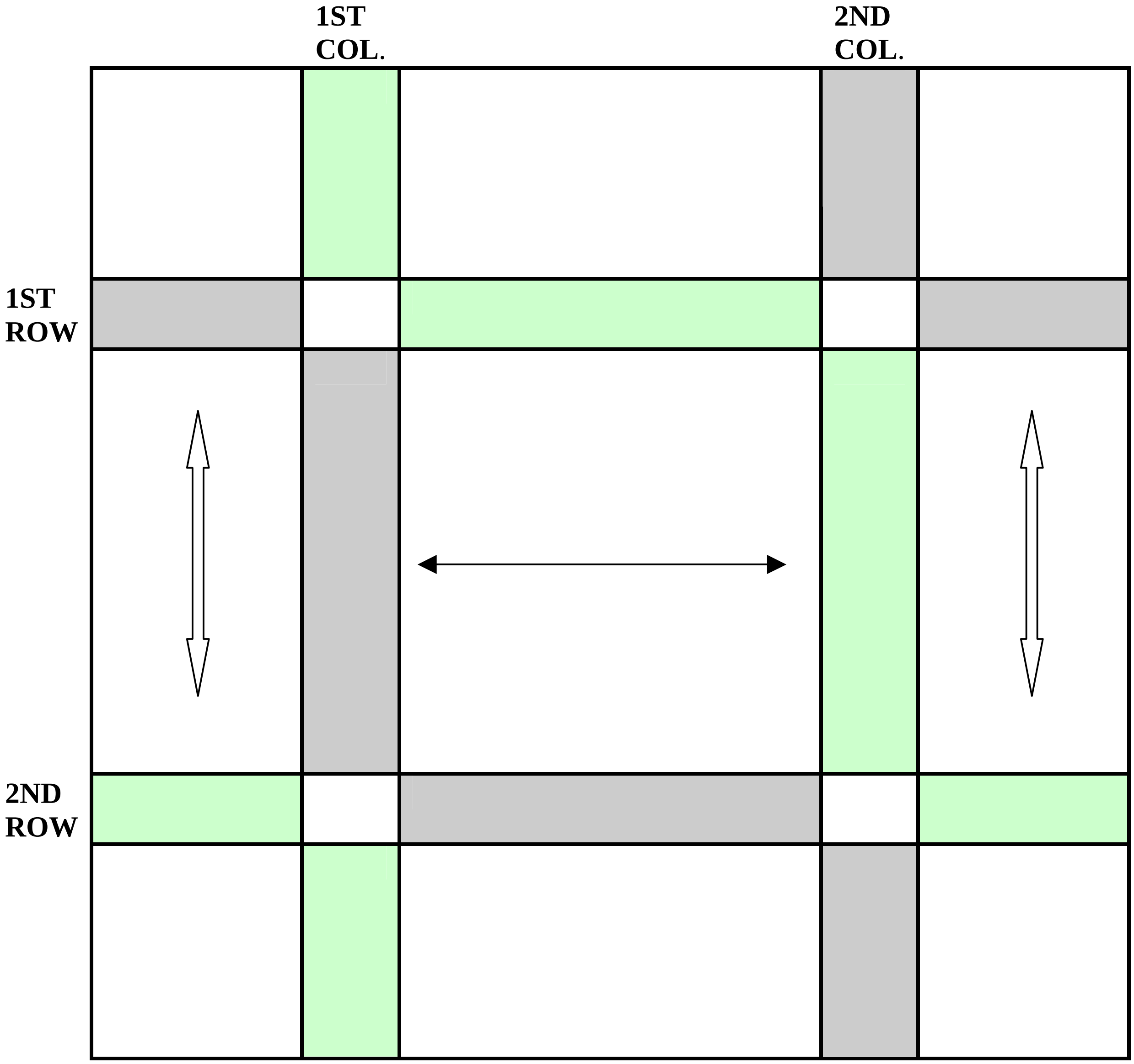
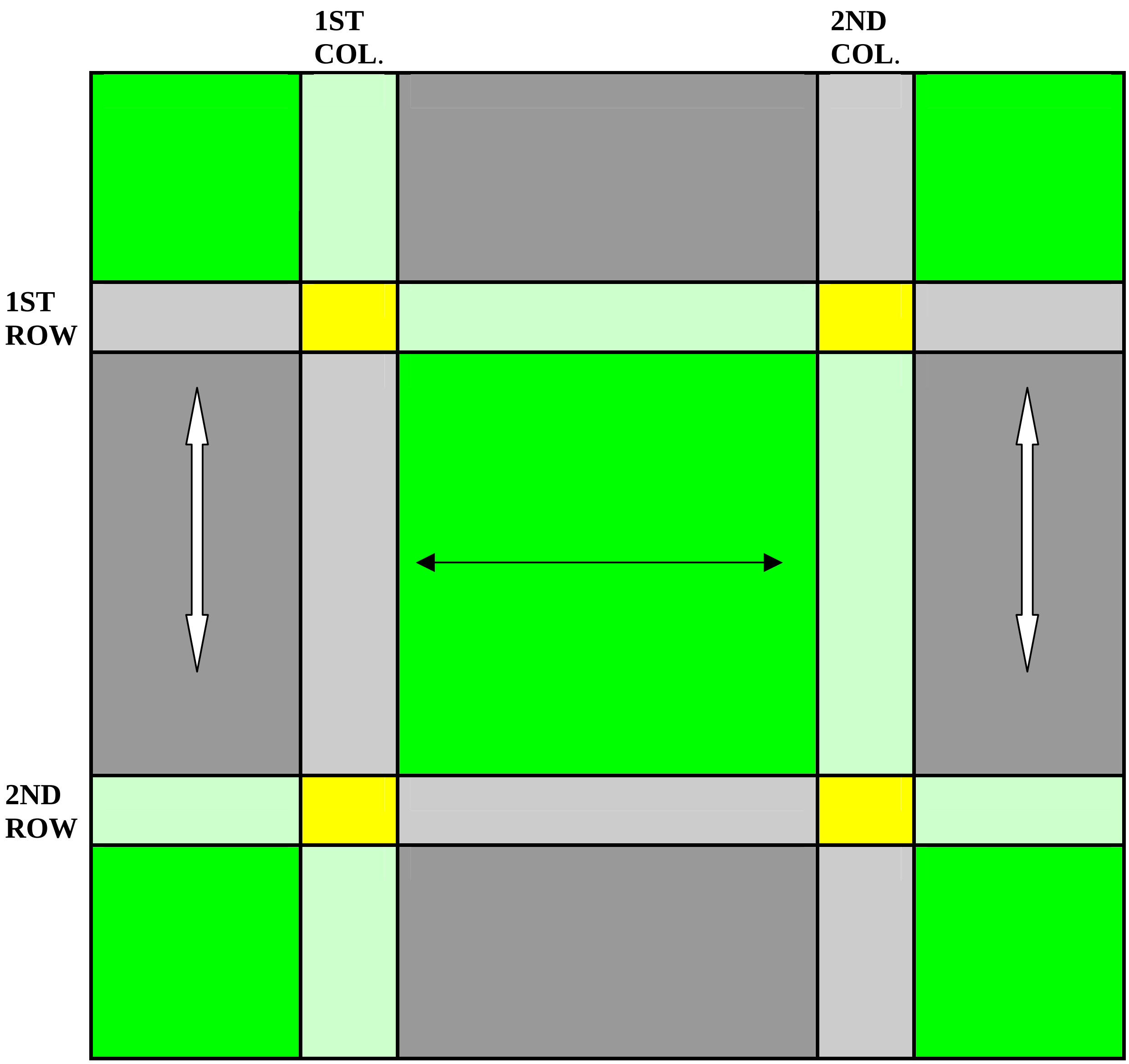


Figure 3.7: Total mapping (before final 4 exchanges)



Exchange

$$\frac{1}{4}\left(2x^2 + x + 2\right) \leftrightarrow \frac{1}{4}\left(2x^2 - x + 2\right)$$

$$\frac{1}{4}x^2 \leftrightarrow \frac{1}{4}x^2 + 1$$

$$\frac{1}{4}x^2 + 1 \leftrightarrow \frac{3}{4}x^2 + 1$$

$$\frac{1}{2}x \leftrightarrow \frac{1}{2}x + 1$$

Figure 3.8: A 14x14 magic square using the previous procedures.

$\theta = 3$				**1ST COL.**							**2ND COL.**				Σ
	1	2	3	4	192	191	190	189	188	187	193	12	13	14	1379
	15	16	17	18	178	177	176	175	174	173	179	26	27	28	1379
	29	30	31	32	164	163	162	161	160	159	165	40	41	42	1379
1ST ROW	154	153	152	46	47	48	**148**	**49**	51	52	53	143	142	141	1379
	140	139	138	137	61	62	63	64	65	66	60	129	128	127	1379
	126	125	124	123	75	76	77	78	79	80	74	115	114	113	1379
	112	111	110	109	89	90	91	92	93	94	88	101	100	99	1379
	98	97	96	**102**	103	104	105	106	107	108	**95**	87	86	85	1379
	84	83	82	81	117	118	119	120	121	122	116	73	72	71	1379
	70	69	68	67	131	132	133	134	135	136	130	59	58	57	1379
2ND ROW	43	44	45	144	150	149	**50**	147	146	145	151	54	55	56	1379
	155	156	157	158	38	37	36	35	34	33	39	166	167	168	1379
	169	170	171	172	24	23	22	21	20	19	25	180	181	182	1379
	183	184	185	186	10	9	**7**	**8**	6	5	11	194	195	196	1379
Σ	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	

A more general solution to the remaining two rows and two columns, excluding diagonal values.

AFTER FILLING THE MAJOR 1-2-1 PATTERN AND REMAINING FOUR DIAGONALS, WE ARE READY TO FILL IN THE REMAINING TWO ROWS / COLUMN'S 1-2-1.

You have an option of completing either the column or row operations first. Column operations are shown first only as a matter of choice.

Remaining 2-Column Operations

Step 1:

Have one column keep its non-diagonal values increasing in order and have the other column change its non-diagonal values to decreasing in order.

This step will give each row involved a sum of (x^2+1) for its remaining two values.
Also, note that for each row, prior to the reversal, the value in column 2 will always exceed that of column 1 by $\frac{1}{2}x$.

Step 2:

Half of the non-diagonal values must be swapped between the two columns.
Swap the central 2-x block.

After this step, the sum of column 2 will exceed column 1 by x units. Since we will not swap one of the diagonals, one additional swap will be required.

Step 3:

Exchange: $\frac{1}{4}(2x^2+x+2)$ with $\frac{1}{4}(2x^2-x+2)$ This step will equalize the two columns.

***** After one of the columns is reversed, these are the only two non-diagonal values on the same row differing by $\frac{1}{2}x$.

<u>Remaining 2-Row Operations</u>

Step 1:
Have one row keep its values increasing in order and have the other row decreasing in order except for the two diagonal values. This will make the columns pairs sum to $\left(x^2+1\right)$.

Step 2:
Half of the non-diagonal values must be swapped between the two rows.
<u>Exchange the two 1-x blocks</u>. Since each row value in the 2nd row exceeds its counterpart in the 1st row by $\frac{1}{2}x^2$, this step will reduces the 2nd row's advantage to x^2.

Step 3:

Three exchanges are needed to balance the rows.

exchange #1: $\frac{1}{4}x^2$ with $\frac{1}{4}x^2+1$ This step aligns row values to be swapped.

exchange #2: $\frac{1}{4}x^2+1$ with $\frac{3}{4}x^2+1$ This step equalizes the two rows.

exchange #3: $\frac{1}{4}x^2-\alpha x$ with $\frac{1}{4}x^2-\alpha x+1$ **<u>OR</u>** $\frac{3}{4}x^2+\alpha x$ with $\frac{3}{4}x^2+\alpha x+1$,

if the section containing $\frac{1}{4}x^2$ is increasing in value.

Exchange #1 made the two affected columns unequal. This step re-equalizes them.
α is a natural number and $\alpha \le \theta$.

OR

$\frac{1}{4}x^2+\beta x$ with $\frac{1}{4}x^2+\beta x+1$ **<u>OR</u>** $\frac{3}{4}x^2-\beta x$ with $\frac{3}{4}x^2-\beta x+1$,

if section containing $\frac{1}{4}x^2$ is decreasing in value and $\theta > 1$.

Exchange #1 made two columns unequal. This step re-equalizes them.
β is a natural number and $\beta < \theta$.

Section 4 6x6 Magic Square — The first 4θ +2 magic square

Figure 4.1 Alphabetic location prior to any rearrangement.

	COL 1	COL 2	COL 3	COL 4	COL 5	COL 6
ROW 1	A	B	C	D	E	F
ROW 2	G	H	I	J	K	L
ROW 3	M	N	O	P	Q	R
ROW 4	S	T	U	V	W	X
ROW 5	Y	Z	AA	AB	AC	AD
ROW 6	AE	AF	AG	AH	AI	AJ

Figure 4.2 Numeric location prior to any rearrangement.

	COL 1	COL 2	COL 3	COL 4	COL 5	COL 6	Σ
ROW 1	1	2	3	4	5	6	21
ROW 2	7	8	9	10	11	12	57
ROW 3	13	14	15	16	17	18	93
ROW 4	19	20	21	22	23	24	129
ROW 5	25	26	27	28	29	30	165
ROW 6	31	32	33	34	35	36	201
Σ	96	102	108	114	120	126	

The 6x6 magic square is the smallest magic square of the form, 4θ + 2. The white shaded area is one pattern, the gray shaded area is another pattern and the yellow are diagonals. Diagonals are never moved.

Figure 4.3 Final alphabetic location after rearrangement.

	COL 1	COL 2	COL 3	COL 4	COL 5	COL 6
ROW 1	A	B	AH	AG	AI	F
ROW 2	AD	H	**AB**	**I**	K	Y
ROW 3	X	W	O	P	N	S
ROW 4	R	T	U	V	Q	M
ROW 5	G	Z	**J**	AA	AC	L
ROW 6	AE	AF	**C**	**D**	E	AJ

Figure 4.4 Final numeric location after rearrangement.

	COL 1	COL 2	COL 3	COL 4	COL 5	COL 6	Σ
ROW 1	1	2	34	33	35	6	111
ROW 2	30	8	**28**	**9**	11	25	111
ROW 3	24	23	15	16	14	19	111
ROW 4	18	**20**	21	22	**17**	13	111
ROW 5	7	26	**10**	27	29	12	111
ROW 6	31	32	**3**	**4**	5	36	111
Σ	111	111	111	111	111	111	

Figure 4.5 Algebraic expression for each 6x6 cell.

A	$\frac{1}{4}(4)$
B	$\frac{1}{4}(x+2)$
C	$\frac{1}{4}(2x)$
D	$\frac{1}{4}(2x+4)$
E	$\frac{1}{4}(3x+2)$
F	$\frac{1}{4}(4x)$
G	$\frac{1}{4}(x^2-2x+4)$
H	$\frac{1}{4}(x^2-x+2)$
I	$\frac{1}{4}(x^2)$
J	$\frac{1}{4}(x^2+4)$
K	$\frac{1}{4}(x^2+x+2)$
L	$\frac{1}{4}(x^2+2x)$
M	$\frac{1}{4}(2x^2-4x+4)$
N	$\frac{1}{4}(2x^2-3x+2)$
O	$\frac{1}{4}(2x^2-2x)$
P	$\frac{1}{4}(2x^2-2x+4)$
Q	$\frac{1}{4}(2x^2-x+2)$
R	$\frac{1}{4}(2x^2)$

S	$\frac{1}{4}(2x^2+4)$
T	$\frac{1}{4}(2x^2+x+2)$
U	$\frac{1}{4}(2x^2+2x)$
V	$\frac{1}{4}(2x^2+2x+4)$
W	$\frac{1}{4}(2x^2+3x+2)$
X	$\frac{1}{4}(2x^2+4x)$
Y	$\frac{1}{4}(3x^2-2x+4)$
Z	$\frac{1}{4}(3x^2-x+2)$
AA	$\frac{1}{4}(3x^2)$
AB	$\frac{1}{4}(3x^2+4)$
AC	$\frac{1}{4}(3x^2+x+2)$
AD	$\frac{1}{4}(3x^2+2x)$
AE	$\frac{1}{4}(4x^2-4x+4)$
AF	$\frac{1}{4}(4x^2-3x+2)$
AG	$\frac{1}{4}(4x^2-2x)$
AH	$\frac{1}{4}(4x^2-2x+4)$
AI	$\frac{1}{4}(4x^2-x+2)$
AJ	$\frac{1}{4}(4x^2)$

Algebraic expressions are not reduced to show the pattern.

Note that the expressions in the second group are $\frac{1}{2}x^2$ larger than the first group.

Figure 4.6 Simplified algebraic expressions for each 6x6 cell.

A	1
B	$\frac{1}{4}x+\frac{1}{2}$
C	$\frac{1}{2}x$
D	$\frac{1}{2}x+1$
E	$\frac{3}{4}x+\frac{1}{2}$
F	x
G	$\frac{1}{4}x^2-\frac{1}{2}x+1$
H	$\frac{1}{4}x^2-\frac{1}{4}x+\frac{1}{2}$
I	$\frac{1}{4}x^2$
J	$\frac{1}{4}x^2+1$
K	$\frac{1}{4}x^2+\frac{1}{4}x+\frac{1}{2}$
L	$\frac{1}{4}x^2+\frac{1}{2}x$
M	$\frac{1}{2}x^2-x+1$
N	$\frac{1}{2}x^2-\frac{3}{4}x+\frac{1}{2}$
O	$\frac{1}{2}x^2-\frac{1}{2}x$
P	$\frac{1}{2}x^2-\frac{1}{2}x+1$
Q	$\frac{1}{2}x^2-\frac{1}{4}x+\frac{1}{2}$
R	$\frac{1}{2}x^2$

S	$\frac{1}{2}x^2+1$
T	$\frac{1}{2}x^2+\frac{1}{4}x+\frac{1}{2}$
U	$\frac{1}{2}x^2+\frac{1}{2}x$
V	$\frac{1}{2}x^2+\frac{1}{2}x+1$
W	$\frac{1}{2}x^2+\frac{3}{4}x+\frac{1}{2}$
X	$\frac{1}{2}x^2+x$
Y	$\frac{3}{4}x^2-\frac{1}{2}x+1$
Z	$\frac{3}{4}x^2-\frac{1}{4}x+\frac{1}{2}$
AA	$\frac{3}{4}x^2$
AB	$\frac{3}{4}x^2+1$
AC	$\frac{3}{4}x^2+\frac{1}{4}x+\frac{1}{2}$
AD	$\frac{3}{4}x^2+\frac{1}{2}x$
AE	x^2-x+1
AF	$x^2-\frac{3}{4}x+\frac{1}{2}$
AG	$x^2-\frac{1}{2}x$
AH	$x^2-\frac{1}{2}x+1$
AI	$x^2-\frac{1}{4}x+\frac{1}{2}$
AJ	x^2

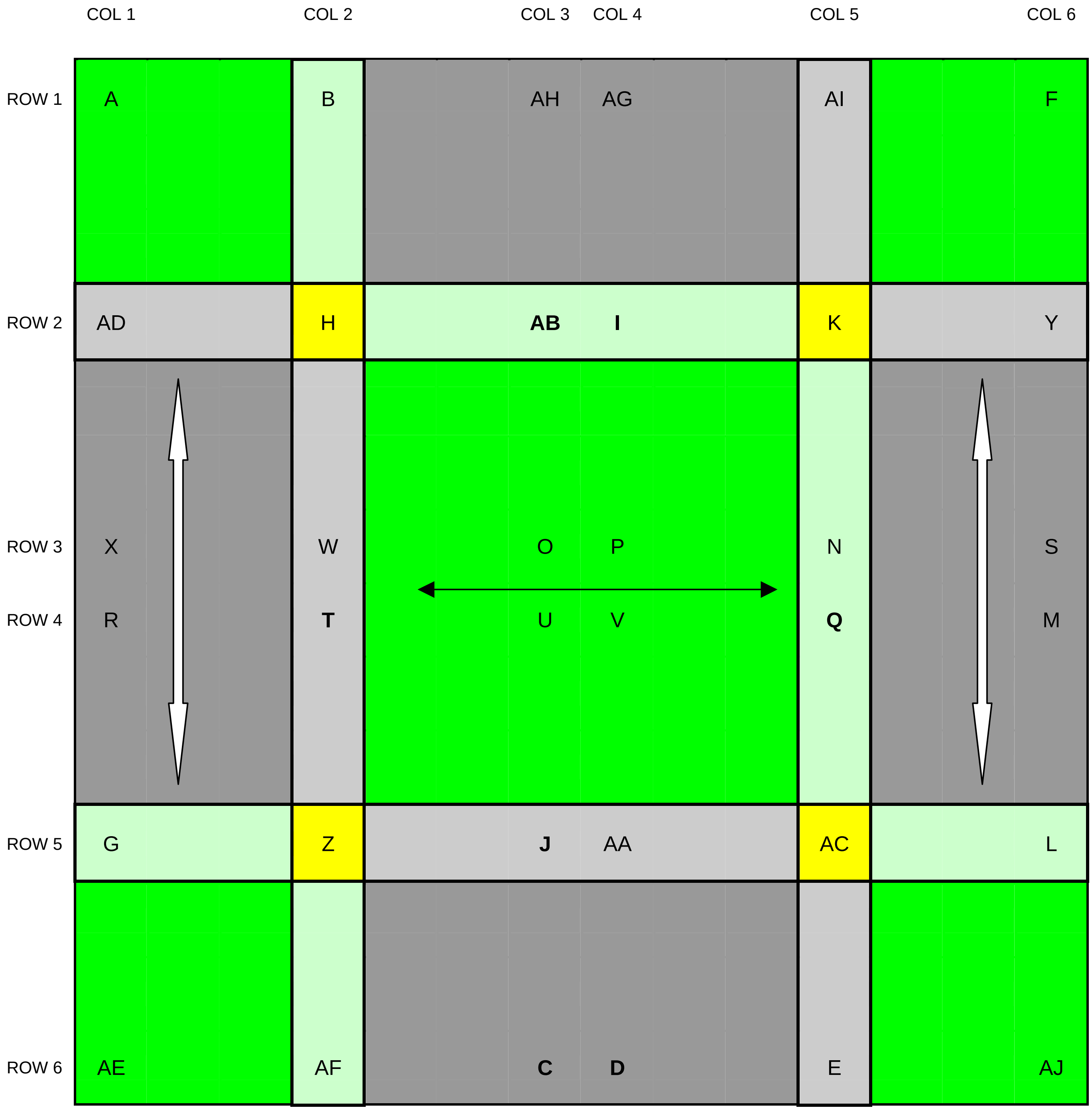
Section 5 General Singly Even Magic Square As Developed from the 6x6 Magic Square

Figure 1 shows all 36 elements of a 6x6 magic square. As the table is expanded for larger singly even magic squares, the algebraic representation of the labeled locations will remain the same. As the table is expanded, new entries will be equally spaced as shown in figure 7.

Figure 5.1 Alphabetic location prior to any rearrangement of a general singly even magic square.

	COL 1	COL 2	COL 3	COL 4	COL 5	COL 6
ROW 1	A	B	**C**	**D**	E	F
ROW 2	G	H	**I**	**J**	K	L
ROW 3	M	N	O	P	**Q**	R
ROW 4	S	**T**	U	V	W	X
ROW 5	Y	Z	AA	**AB**	AC	AD
ROW 6	AE	AF	AG	AH	AI	AJ

Figure 5.2 Alphabetic location after rearrangement of a general singly even magic square.



Columns 2 & 5 along with rows 2 & 5 form a pattern. Reverse the order of row 5 and column 5 (light gray – shows reverse order and light green shows the original order). Swap the parts as shown on figure 8. Five cells, in bold, are repositioned. (I↔J, J↔AB, (Q↔T)

The rest of the magic square forms a 1-2-1 pattern as in a doubly even magic square (dark gray – shows reverse order and dark green shows the original order).
Only two cells, in bold, need to be repositioned. C↔D)

All remaining cells will follow their respective pattern regardless of how large you choose to make the singly even magic square.

Figure 5.3: Total mapping (after final 4 exchanges)

$\theta = 3$				1ST COL.							2ND COL.				Σ
	1	2	3	4	192	191	190	189	188	187	193	12	13	14	1379
	15	16	17	18	178	177	176	175	174	173	179	26	27	28	1379
	29	30	31	32	164	163	162	161	160	159	165	40	41	42	1379
1ST ROW	154	153	152	46	47	48	**148**	**49**	51	52	53	143	142	141	1379
	140	139	138	137	61	62	63	64	65	66	60	129	128	127	1379
	126	125	124	123	75	76	77	78	79	80	74	115	114	113	1379
	112	111	110	109	89	90	91	92	93	94	88	101	100	99	1379
	98	97	96	**102**	103	104	105	106	107	108	**95**	87	86	85	1379
	84	83	82	81	117	118	119	120	121	122	116	73	72	71	1379
	70	69	68	67	131	132	133	134	135	136	130	59	58	57	1379
2ND ROW	43	44	45	144	150	149	**50**	147	146	145	151	54	55	56	1379
	155	156	157	158	38	37	36	35	34	33	39	166	167	168	1379
	169	170	171	172	24	23	22	21	20	19	25	180	181	182	1379
	183	184	185	186	10	9	**7**	**8**	6	5	11	194	195	196	1379
Σ	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	1379	

The four exchanges are:

$\frac{1}{4}(2x^2 + x + 2) \leftrightarrow \frac{1}{4}(2x^2 - x + 2)$ **(T↔Q)**

$\frac{1}{4}x^2 \leftrightarrow \frac{1}{4}x^2 + 1$ **(I↔J)**

$\frac{1}{4}x^2 + 1 \leftrightarrow \frac{3}{4}x^2 + 1$ **(J↔AB)**

$\frac{1}{2}x \leftrightarrow \frac{1}{2}x + 1$ **(C↔D)**

Figure 5.4: Linear multiple of a Magic Square.

5	8	11	14	578	575	572	569	566	563	581	38	41	44	4165
47	50	53	56	536	533	530	527	524	521	539	80	83	86	4165
89	92	95	98	494	491	488	485	482	479	497	122	125	128	4165
464	461	458	140	143	146	**446**	**149**	155	158	161	431	428	425	4165
422	419	416	413	185	188	191	194	197	200	182	389	386	383	4165
380	377	374	371	227	230	233	236	239	242	224	347	344	341	4165
338	335	332	329	269	272	275	278	281	284	266	305	302	299	4165
296	293	290	**308**	311	314	317	320	323	326	**287**	263	260	257	4165
254	251	248	245	353	356	359	362	365	368	350	221	218	215	4165
212	209	206	203	395	398	401	404	407	410	392	179	176	173	4165
131	134	137	434	452	449	**152**	443	440	437	455	164	167	170	4165
467	470	473	476	116	113	110	107	104	101	119	500	503	506	4165
509	512	515	518	74	71	68	65	62	59	77	542	545	548	4165
551	554	557	560	32	29	**23**	**26**	20	17	35	584	587	590	4165
4165	4165	4165	4165	4165	4165	4165	4165	4165	4165	4165	4165	4165	4165	

Magic squares may be multiplied by a linear value, in this case, each value was replaced with 3x + 2 times its value.